

# Field-Driven Partition Rearrangement at Quantum Phase Transitions and Local Coarse-Graining Failure from Informational Saturation

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April 2026

## Abstract

We present two results in quantum information theory motivated by the emergent spacetime program.

First, we prove that for any finite-dimensional Hamiltonian  $H(\phi)$  depending on a scalar parameter  $\phi$ , the bipartition maximizing the total correlation  $I_{\text{tot}}$  can rearrange when the correlation structure of the ground state changes qualitatively (Theorem 3.1). We test this across three models via exhaustive enumeration of all equal bipartitions. The  $J_1$ - $J_2$  Heisenberg chain exhibits a genuine rearrangement: the optimal partition switches from sublattice to dimer-splitting near the Majumdar-Ghosh point ( $J_2/J_1 = 0.5$ ) at  $N = 8$  and  $N = 10$ , tracking the reorganization from Néel to dimer correlations. Two negative results establish the mechanism's boundaries: the transverse-field Ising model and the antiferromagnetic XXZ chain show no rearrangement at their respective QPTs, because their correlation structure changes in strength but not in type.

Second, we derive a local coarse-graining failure (LCGF) result: within holographic settings, when  $I_{\text{tot}}$  saturates its upper bound in a spatial region, the modular Berry curvature vanishes (Proposition 4.1), and no classical geometry can be extracted by the class of coarse-graining maps considered here (Theorem 4.3). Applied to gravitational collapse, this provides a derivation of the black hole interior as a region where emergent geometry is undefined within this reconstruction scheme.

The two results probe complementary regimes of qualitative change in informational structure: Part I addresses when the optimal partition reorganizes; Part II addresses when no partition supports geometry at all.

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# 1 Introduction

The emergent spacetime program—the hypothesis that spacetime geometry arises from quantum information structure—has produced several precise results. The Ryu–Takayanagi formula [1] equates bulk geometry with boundary entanglement entropy. Van Raamsdonk [2] showed that reducing entanglement disconnects bulk spacetime. Jacobson [3, 4] derived the Einstein equation from thermodynamic and entanglement equilibrium assumptions. Faulkner et al. [5] derived the linearized Einstein equation from the entanglement first law within AdS/CFT. MERA tensor networks [6, 7] reproduce spatial geometry from entanglement structure.

Two open questions in this program motivate the present work:

**(Q1) Partition selection.** If spacetime emerges from the correlation structure of a Hilbert space, which decomposition into subsystems generates the physical geometry? Quantum mereology [8, 9, 10, 11] has proposed several criteria. Zanardi et al. [12] recently showed that the preferred structure undergoes sharp transitions—mereological phase transitions—as Hamiltonian parameters are varied. But the question of which physical conditions produce such transitions has not been systematically investigated.

**(Q2) Geometric breakdown.** If geometry emerges from structured entanglement, what happens when entanglement becomes featureless? Czech et al. [13] showed that modular Berry curvature encodes the bulk Riemann tensor in holographic settings, but the consequences of Berry curvature vanishing have not been developed.

This paper contributes one result to each question, plus a systematic negative result that sharpens our understanding of (Q1).

For (Q1), we prove that  $I_{\text{tot}}$ -maximizing partitions can rearrange when the ground state’s correlation structure reorganizes qualitatively (Theorem 3.1). We test this in three models. The  $J_1$ - $J_2$  Heisenberg chain provides a positive example: the optimal partition switches from sublattice to dimer-splitting, tracking the Néel-to-dimerized correlation reorganization. The transverse-field Ising model and antiferromagnetic XXZ chain provide negative examples: their QPTs change correlation *strength* without changing which partition type captures the dominant correlations, and no rearrangement occurs. Together, these results identify a necessary condition: the mechanism requires the QPT to reorganize which degrees of freedom are correlated, not merely how strongly.

For (Q2), we show that informational saturation ( $I_{\text{tot}} \rightarrow I_{\text{max}}$ ) produces local coarse-graining failure (Theorem 4.3), providing a derivation of black hole interior undefinability within holographic settings.

The two results probe complementary regimes. Part I concerns *reorganization*: the optimal subsystem decomposition rearranges because the correlation structure shifts at a QPT. Part II concerns *breakdown*: correlations saturate and become featureless, and no partition supports geometry. In both cases,  $I_{\text{tot}}$  is the diagnostic variable—its maximizer shifts in Part I, its saturation triggers failure in Part II.

In Section 6, we discuss implications within the Seam-Fold-Bulk (SFB) framework [14], clearly separating established results from speculative extensions.

## 2 Preliminaries

### 2.1 Total Correlation

**Definition 2.1** (Total Correlation). For a state  $\rho \in \mathcal{D}(\mathcal{H})$  on  $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$  with  $\dim(\mathcal{H}_i) = d$ , the total correlation is

$$I_{\text{tot}}(\rho; P) = \sum_{i=1}^n S(\rho_i) - S(\rho) = D\left(\rho \parallel \bigotimes_{i=1}^n \rho_i\right), \quad (1)$$

where  $S(\cdot) = -\text{Tr}(\cdot \log \cdot)$  is the von Neumann entropy and  $D(\cdot \parallel \cdot)$  the quantum relative entropy.

**Proposition 2.2** (Properties of  $I_{\text{tot}}$ ). (i)  $I_{\text{tot}} \geq 0$ , with equality iff  $\rho = \bigotimes_i \rho_i$ . (ii)  $I_{\text{tot}} \leq (n-1) \log d \equiv I_{\text{max}}$ . (iii)  $I_{\text{tot}}(\Lambda(\rho)) \leq I_{\text{tot}}(\rho)$  for any product of local CPTP maps  $\Lambda = \bigotimes_i \Lambda_i$ . (iv)  $I_{\text{tot}}$  is invariant under local unitaries.

*Proof.* (i), (iii): non-negativity and data processing inequality for quantum relative entropy [15, 16]. (ii):  $S(\rho_i) \leq \log d$ ,  $S(\rho) \geq 0$ . (iv): unitary invariance of von Neumann entropy.  $\square$

### 2.2 Coarse-Graining and Geometric Emergence

**Definition 2.3** (Coarse-Graining Maps). A family of CPTP maps  $\mathcal{C}_\ell : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{G}$ , where  $\mathcal{G} = \{(g_{\mu\nu}, T_{\mu\nu}, \Lambda_{\text{eff}})\}$ . Classical spacetime exists if  $\mathcal{C}_\ell$  converges to stable fixed points. The geometric convergence norm  $\Delta(\ell, \tau) = \|\mathcal{C}_\ell[\rho(\tau)] - G_{\text{cl}}\|_{\mathcal{G}}$  measures deviation from semiclassical geometry.

## 3 Part I: Field-Driven Partition Rearrangement

### 3.1 Main Result

Let  $H(\phi)$  be a one-parameter family of Hamiltonians on a finite-dimensional Hilbert space  $\mathcal{H}$ , with ground state  $|\Psi_0(\phi)\rangle$ . Define  $I_\theta(\phi) = I_{\text{tot}}(|\Psi_0(\phi)\rangle\langle\Psi_0(\phi)|; \theta)$  for bipartition  $P_\theta$ , and  $\theta^*(\phi) = \arg \max_\theta I_\theta(\phi)$ .

**Theorem 3.1** (Partition Rearrangement). Let  $\{P_\theta\}$  be a finite collection of bipartitions. Suppose there exist  $\theta_1, \theta_2$  such that  $I_{\theta_1}(\phi) > I_{\theta_2}(\phi)$  for  $\phi < \phi_c$  and  $I_{\theta_2}(\phi) > I_{\theta_1}(\phi)$  for  $\phi > \phi_c$ . Then  $\theta^*(\phi)$  changes value at some  $\phi^* \in [\phi_c^-, \phi_c^+]$ .

*Proof.* The argmax over a finite discrete set changes value when curves cross. By hypothesis, a crossing occurs.  $\square$

**Remark 3.2** (Content). The mathematical content is modest. The physical question is: which QPTs satisfy conditions (i)–(ii)? The numerical results below show that the answer is nontrivial—some QPTs produce rearrangements and others do not, depending on whether the correlation type changes or merely its strength.

### 3.2 Positive Result: $J_1$ - $J_2$ Heisenberg Chain

The  $J_1$ - $J_2$  Heisenberg chain with antiferromagnetic couplings:

$$H(J_2) = J_1 \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \sum_{i=1}^{N-2} \vec{S}_i \cdot \vec{S}_{i+2}, \quad (2)$$

with  $J_1 = 1$ ,  $\vec{S}_i = \vec{\sigma}_i/2$ , and open boundary conditions. The model has two phases: a gapless Néel phase for  $J_2/J_1 \lesssim 0.2412$  and a gapped dimerized phase for  $J_2/J_1 \gtrsim 0.2412$  [18, 19]. At  $J_2/J_1 = 0.5$ , the Majumdar-Ghosh (MG) point [17], the exact ground state is a product of nearest-neighbor singlets.

The two phases have qualitatively different correlation structures: long-range staggered (Néel) correlations  $\langle \vec{S}_i \cdot \vec{S}_j \rangle \sim (-1)^{|i-j|}$  in the Néel phase, and short-range dimer correlations  $\langle (\vec{S}_{2k} \cdot \vec{S}_{2k+1})(\vec{S}_{2l} \cdot \vec{S}_{2l+1}) \rangle$  in the dimerized phase. This reorganization of correlation *type*—not merely strength—is the condition required by Theorem 3.1.

**Method.** For  $N \in \{8, 10\}$ , we compute  $I_{\text{tot}}$  for every equal bipartition ( $\binom{N}{N/2} = 70$  and 252 respectively) via exact diagonalization, and identify the maximizer at each  $J_2$ .

#### Results.

**1. Sublattice dominance in the Néel regime.** For  $J_2 < 0.5$  at both system sizes, the  $I_{\text{tot}}$ -maximizing bipartition is the sublattice decomposition (even sites vs. odd sites). This reflects the dominance of staggered correlations: the sublattice partition maximally separates antiferromagnetically correlated partners.

**2. Rearrangement at the Majumdar-Ghosh point.** At  $J_2 \approx 0.5$ , the sublattice partition loses optimality. Non-standard bipartitions—those that split dimer pairs  $(2k, 2k+1)$  between the two subsystems—become optimal. At the MG point itself ( $J_2 = 0.5$ ), all bipartitions that split every dimer pair achieve the same maximal  $I_{\text{tot}} = N$  bits. For  $J_2 > 0.5$ , dimer-splitting partitions strictly dominate.

**3. Physical mechanism.** Néel correlations are long-range and staggered: the sublattice partition, which groups every other site, captures these maximally. Dimer correlations are short-range and paired: partitions that split each dimer pair capture these maximally. The partition rearrangement occurs when dimer correlations become strong enough to dominate the optimal partition, which happens near  $J_2/J_1 = 0.5$  for these system sizes.

	$N = 8$	$N = 10$
Bipartitions enumerated	70	252
$J_2$ at sublattice loss	$\approx 0.50$	$\approx 0.50$
$I_{\text{tot}}^{\text{max}}$ at MG point (bits)	8.00	10.00
Optimal before $J_2 = 0.5$	sublattice (even odd)	sublattice (even odd)
Optimal after $J_2 = 0.5$	dimer-splitting	dimer-splitting

Table 1: Partition rearrangement in the  $J_1$ - $J_2$  chain. The sublattice partition loses optimality near the Majumdar-Ghosh point at both system sizes. Runtime:  $< 2$  min per system size (Python/NumPy/SciPy).

**Remark 3.3** (Rearrangement point vs. QPT). *The rearrangement occurs at  $J_2 \approx 0.5$  (the MG point), not at  $J_2 \approx 0.2412$  (the actual QPT). The QPT opens a gap and initiates*

dimerization, but the dimer correlations do not dominate over residual Néel correlations until  $J_2$  is well into the dimerized phase. Whether the rearrangement point converges toward the QPT as  $N \rightarrow \infty$  is an open question requiring DMRG at  $N = 50$ – $100$ .

### 3.3 Negative Results: TFIM and Antiferromagnetic XXZ

Two models show no partition rearrangement, establishing the mechanism’s boundaries.

#### Transverse-field Ising model.

$$H(\Gamma) = -J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z - \Gamma \sum_{i=1}^N \sigma_i^x, \quad J = 1. \quad (3)$$

The QPT at  $\Gamma_c = 1$  separates an ordered ( $z$ -polarized) phase from a disordered ( $x$ -polarized) phase. Exhaustive enumeration at  $N = 8$  and  $N = 10$  over  $\Gamma \in [0.1, 3.0]$  shows that the sublattice partition is the  $I_{\text{tot}}$  maximizer at every parameter value. No rearrangement occurs.

#### Antiferromagnetic XXZ chain.

$$H(\Delta) = J \sum_{i=1}^{N-1} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z), \quad J = 1. \quad (4)$$

The BKT QPT at  $\Delta = 1$  separates a gapless XY phase from a gapped Néel phase. Exhaustive enumeration at  $N = 10$  over  $\Delta \in [0, 2]$  shows sublattice dominance at every parameter value. No rearrangement occurs.

**Remark 3.4** (Sign convention). *The antiferromagnetic XXZ model has  $H = +J \sum \dots$  with  $J > 0$  (positive sign), which penalizes aligned spins and produces the BKT transition at  $\Delta = 1$ . The ferromagnetic convention  $H = -J \sum \dots$  used in some previous work [14] produces a trivial transition from easy-plane to easy-axis ferromagnet within the maximal-spin sector, with the ground state collapsing to a product state for  $\Delta > 1$ . This is not a BKT transition.*

**Why these models show no rearrangement.** In both the TFIM and antiferromagnetic XXZ, the QPT changes the *strength* and *range* of correlations but not their *spatial organization*. In the TFIM, nearest-neighbor  $\sigma^z \sigma^z$  correlations dominate in both phases; the transverse field only modulates their amplitude. In the antiferromagnetic XXZ, staggered correlations dominate on both sides of the BKT transition. In neither case do two different partition types capture qualitatively different correlation structures—the condition required by Theorem 3.1.

By contrast, the  $J_1$ - $J_2$  chain has two competing correlation types: long-range staggered (captured by sublattice) and short-range paired (captured by dimer-splitting). The QPT reorganizes which type dominates, producing the conditions of the theorem.

**Remark 3.5** (Necessary condition). *The positive and negative results together suggest a necessary condition for field-driven partition rearrangement: the QPT must reorganize the type of dominant correlations, not merely their strength. Specifically, the two phases must have ground states whose correlation structures are maximally captured by different partition classes. This is a physically meaningful criterion that goes beyond the formal conditions of Theorem 3.1.*

## 4 Part II: Local Coarse-Graining Failure from Informational Saturation

### 4.1 Main Result

**Proposition 4.1** (Saturated States are Geometrically Inert). *If  $\rho_i = \mathbb{I}/d$  for all subsystems  $\mathcal{H}_i$  in partition  $P$ , then the modular Berry curvature  $F_{\mu\nu}(\lambda)$  vanishes identically.*

*Proof.* When  $\rho_i = \mathbb{I}/d$ , the modular Hamiltonian  $K_{A_\lambda} = -\log \rho_{A_\lambda} \propto \mathbb{I}$ , independent of  $\lambda$ . Since  $\partial_\mu K_{A_\lambda} = 0$ , the Berry connection is pure gauge and  $F_{\mu\nu} = 0$ .  $\square$

**Definition 4.2** (Local Coarse-Graining Failure). *A region  $\mathcal{R}$  exhibits LCGF if  $\Delta(\ell, \tau)|_{\mathcal{R}} \gg 1$  while  $\Delta(\ell, \tau)|_{\bar{\mathcal{R}}} \ll 1$ .*

**Theorem 4.3** (LCGF from Informational Saturation). *(Within holographic settings.)*

*Suppose  $I_{\text{tot}}(\rho_R) \rightarrow I_{\text{max}}$  in a spatial region  $R$ . Then: (i) All reduced states satisfy  $\rho_i \rightarrow \mathbb{I}/d$ . (ii) The modular Berry curvature vanishes in  $R$  (Proposition 4.1). (iii) Within AdS/CFT—at leading order in  $1/N$ , within the code subspace, and for holographic CFTs with semiclassical gravity duals—the Berry curvature equals the Riemann tensor at the corresponding RT surface [13]. Therefore no Riemann tensor can be extracted from  $R$  by the class of coarse-graining maps considered here, and  $\Delta|_R \rightarrow \infty$ .*

**Conjecture 4.4** (Geometric Inertness Outside AdS/CFT). *For any coarse-graining maps satisfying Definition 2.3, vanishing modular Berry curvature implies  $\Delta \rightarrow \infty$ .*

### 4.2 Application to Black Holes

Applied to gravitational collapse, Theorem 4.3 yields the picture of a black hole interior as a region where informational saturation has destroyed the structured entanglement required for emergent geometry. The boundary where  $\Delta$  transitions from  $\ll 1$  to  $\gg 1$  corresponds to the event horizon.

**Structured vs. featureless entanglement.** The thermofield double  $|TFD\rangle$  carries maximal entanglement yet is dual to connected geometry. This does not contradict Theorem 4.3: the TFD has Boltzmann-weighted entanglement with nonvanishing Berry curvature. Maximally mixed states  $\rho_i \approx \mathbb{I}/d$  carry featureless entanglement with vanishing Berry curvature.

**Information preservation.** Under unitary dynamics,  $S(\rho(\tau))$  is conserved. Information is reorganized from interior to boundary degrees of freedom. The resulting encoding map is non-isometric, consistent with Akers et al. [23].

**Relation to existing results.** Theorem 4.3 provides an independent route to interior undefinability consistent with Engelhardt–Wall [22], Akers et al. [23], and Leutheusser–Liu [24]. The claim is the weaker, defensible one: interior geometry is *undefined within this reconstruction scheme*, not absolutely non-existent.

### 4.3 Toy Model Verification

(1) **Random tensor networks** [20]: below threshold ( $s_{\text{bulk}} < \log D$ ),  $\Delta \ll 1$ ; above threshold,  $\rho_\partial \rightarrow \mathbb{I}/D^{|\mathcal{A}|}$ , Berry curvature vanishes,  $\Delta \rightarrow \infty$ .

(2) **Random brickwork circuits** [21]: after  $k \sim \log n$  layers,  $I_{\text{tot}} \rightarrow I_{\text{max}}$ , the entanglement spectrum flattens, and  $\Delta \rightarrow \infty$ .

## 5 Open Problems

### For Part I

- 1. Convergence of rearrangement point.** In the  $J_1$ - $J_2$  chain, does the rearrangement point converge to the QPT at  $J_2/J_1 \approx 0.2412$  as  $N \rightarrow \infty$ ? DMRG at  $N = 50$ – $100$  would provide evidence.
- 2. Additional models with correlation-type reorganization.** The necessary condition of Remark 3.5 predicts rearrangement in other models with competing correlation types: the Shastry-Sutherland model (Néel vs. plaquette), the Kitaev-Heisenberg model (zigzag vs. stripy), and frustrated magnets with competing orders.
- 3. Relation to other partition criteria.** How does  $I_{\text{tot}}$ -maximization compare with Zanardi et al.’s algebra susceptibility [12] and Carroll-Singh’s quasi-classicality criterion [8]?
- 4. Thermodynamic-limit sharpening.** Does the crossover become a non-analytic transition as  $N \rightarrow \infty$ ?

### For Part II

- 5. Geometric inertness outside AdS/CFT.** Proving Conjecture 4.4 without holographic assumptions.
- 6. Observer dependence at horizons** [25, 26].

## 6 Discussion: Implications for Emergent Spacetime

**Everything in this section beyond the established results is speculative.**

The Seam-Fold-Bulk (SFB) framework [14] proposes emergent spacetime from  $I_{\text{tot}}$  dynamics. The present results bear on it as follows:

**Part I implications.** The partition rearrangement result shows that  $I_{\text{tot}}$ -maximizing partitions can respond to changes in Hamiltonian parameters, which is a prerequisite for the SFB mechanism of quintessence-driven partition selection. However, the negative results show this does not happen generically—it requires correlation-type reorganization. Whether cosmological quintessence dynamics produce this type of reorganization is unknown and cannot be addressed without solving the continuum-limit problem (connecting lattice mereology to field-theoretic spacetime).

**Part II implications.** Theorem 4.3 applied within SFB yields the picture of black holes as local Seam regions embedded in a global Bulk.

**Limitations of the SFB framework.** Three foundational gaps remain: (a)  $I_{\text{tot}}$ -maximization is assumed, not derived; (b) recovery of the Einstein equation from coarse-graining fixed-point structure is not demonstrated (cf. Jacobson [3, 4], Faulkner et al. [5]); (c) observational predictions are qualitative.

**Erratum.** Previous versions of the SFB framework [14] used a ferromagnetic XXZ Hamiltonian ( $H = -J \sum \dots$ ,  $J > 0$ ) while claiming a BKT quantum phase transition at  $\Delta = 1$ . The ferromagnetic model has a trivial transition within the maximal-spin sector at this point, not a BKT transition. The BKT transition occurs in the antiferromagnetic

model ( $H = +J\sum \dots$ ), which shows no partition rearrangement. The present paper corrects this by using the  $J_1$ - $J_2$  chain, which exhibits a genuine rearrangement driven by a physically meaningful correlation reorganization.

## 7 Conclusion

**Result 1: Partition rearrangement.** Theorem 3.1 proves that  $I_{\text{tot}}$ -maximizing partitions can rearrange at quantum phase transitions. The  $J_1$ - $J_2$  Heisenberg chain confirms this: the optimal partition switches from sublattice to dimer-splitting near the Majumdar-Ghosh point at  $N = 8$  and  $N = 10$ , tracking the reorganization from Néel to dimer correlations. Negative results in the TFIM and antiferromagnetic XXZ establish that the mechanism requires reorganization of correlation *type*, not merely strength.

**Result 2: LCGF from informational saturation.** Theorem 4.3 shows that when  $I_{\text{tot}}$  saturates, modular Berry curvature vanishes and (within holographic settings) no classical geometry can be extracted by the class of coarse-graining maps considered here.

**Nearest-term extensions:** DMRG scaling of the  $J_1$ - $J_2$  rearrangement to  $N = 50$ – $100$ ; testing additional models with competing correlation types; proving Conjecture 4.4 outside holography.

**Reproducibility.** All simulation code (Python/NumPy/SciPy) is provided as supplementary material.

**Acknowledgments.** The author thanks the theoretical physics community whose foundational work in holography, tensor networks, quantum information theory, and quantum mereology made this work possible.

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